EE 670 : Homework #2 Solution

Prof. Uf Tureli, Stevens Institute of Technology
email/tel/fax: utureli@stevens-tech.edu, 201.216.5603/8246

Note: Cover and Thomas, Q. 3.4,4.2,4.7,4.10

1. Question: *Products.* Let

$$X = \begin{cases} 1, & \text{with probability } \frac{1}{2} \\ 2, & \text{with probability } \frac{1}{4} \\ 3, & \text{with probability } \frac{1}{24} \end{cases} \tag{1}$$

   Let $X_1, X_2, \cdots$ be drawn i.i.d. according to this distribution. Find the limitiing behavior of the product $(X_1 X_2 \cdots X_n)^{\frac{1}{n}}$. **Solution**: Let $P_n = (X_1 X_2 \cdots X_n)^{\frac{1}{n}}$. Then:

$$\log P_n = \frac{1}{n} \sum i = 1^n X_i \to X_i,$$

   with probability 1 by strong law of large numebr. Thus, $Pn \to 2^{E \log X}$ with prob. 1 and $E \log X = \frac{1}{2} \log 1 + \frac{1}{4} \log 2 + \frac{1}{4} \log 3 = \frac{1}{4} \log 6$. $P_n \to 2^{\frac{1}{2} \log 6}$.

2. Question: *Time's arrow.* Let $\{X_i\}_{i=-\infty}^{\infty}$ be a stationary stochastic process. Prove:

$$H(X_0|X_{-1}, X_{-2}, \cdots, X_{-n}) = H(X_0|X_1, X_2, \cdots, X_n)$$

   Solution: By chain rule for entropy:

$$\begin{aligned} H(X_0|X_{-1}, X_{-2}, \cdots, X_{-n}) &= H(X_0, X_{-1}, X_{-2}, \cdots, X_{-n}) - H(X_{-1}, X_{-2}, \cdots, X_{-n}) \\ &= H(X_0, X_1, X_2, \cdots, X_n) - H(X_1, X_2, \cdots, X_n) \\ &= H(X_0|X_1, X_2, \cdots, X_n) \end{aligned}$$

   where the second equation follows from stationarity.

3. Question: *Initial Conditions.* Show for a *stationary* Markov chain that:

$$H(X_0|X_n) \geq H(X_0|X_{n-1}).$$

   Thus, initial conditions become more difficult to recover as the futer X_n unfolds.

   Solution: For a Markov chan, by the data processing theorem, we have:

$$I(X_0; X_{n-1}) \geq I)X_0; X_n).$$

   Therefore,

$$H(X_0) - H(X_0|X_{n-1}) \geq H(X_0) - H(X_0|X_n),$$

   or $H(X_0|X_n)$ increases with n.

4. Question: *Entropy rate of a dog looking for a bone.* A dog walks on the integers possibly reversing direction at each step with probability p=0.1. Let $X_0 = 0$. The first step is equally likely to be positive or negateive. A typical walk might look like this:

$$(X_0, X_1, \cdots) = (0, -1, -2, -3, -4, -3, -2, -1, -, 1, \cdots)$$

(a) Find $H(X_1, X_2, \cdots, X_n)$

Solution: By chain rule:

$$\begin{aligned} H(X_0, X_2, \cdots, X_n) &= \sum i = 0^n H(X_i|X_{i-1}) \\ &= H(X_0) + H(X_1|X_0) + \sum i = 2^n H(X_i|X_{i-1}) \end{aligned}$$

since for $i > 1$, next position only depends on the previous two. (Dog's walk is 2nd order Markov). Since $X_0 = 0$ deterministically, $H(X_0) = 0$ and since first step is equally likely to be positive or negative $H(X_1|X_0) = 1$. For $i > 1$:

$$H(X_i|X_{i-1}, X_{i-2}) = H(0.1, 0.9).$$

Therefore, $H(X_1, X_2, \cdots, X_n) = 1 + (n-1)H(0.1)(0.9)$.

(b) Find entropy rate of this browsing dog.

Solution:

$$\begin{aligned} \frac{H(X_0, X_1, \cdots, X_n)}{n+1} &= \frac{1 + (n-1)H(0.1, 0.9)}{n+1} \\ &\to H(0.1, 0.9) \end{aligned}$$

(c) What is the expected number of steps the dog takes before reversing direction?

Solution: the dog must take at least one step to establish the direction of travel from which it ultimately reverses. Let S be the number of steps taken between reversals, we have:

$$E(S) = \sum s = 1 \infty s(0.9)^{s-1}(0.1) = 10.$$