Midterm for CpE/EE/PEP 345 – Modeling and Simulation
Stevens Institute of Technology
Fall 2003

The midterm is open book/open notes. Total value is 100 points (30% of course grade). All questions are equally weighted. Do any 10 of the 12 question. Do more than 10 for extra credit. Some question can be answered in more than one way. Only one answer is required, but extra credit will be given for identifying and explaining alternate answers.

1. The Hoboken Widget Manufacturing Company uses a 5 step process to manufacture widgets, each performed by a different employee. As it happens, the time to complete each step is independent with an exponentially distributed completion time with a mean $1/m$.. Therblig Associates is helping to refine the process and has assumed that the arrival process at the next stage, the testing stage, is a Poisson process. (a) is this assumption correct? (b) prove or disprove the assertion (c) would it make any difference if the entire manufacturing process were carried out by each employee who could complete the process with an exponentially distributed with a mean completion time of $5/m$?

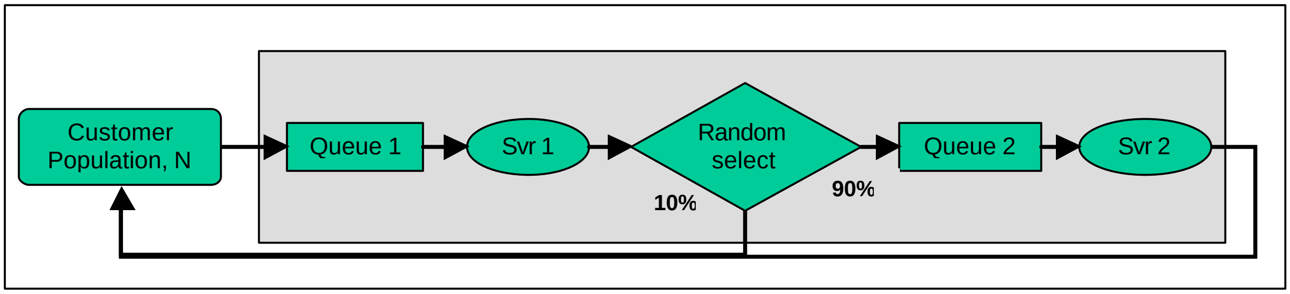
> [2 pts] The assumption is not correct.
> [5 pts] For the arrival process at the testing stage to be Poisson, the interarrival time would have to be exponentially distributed. Although the individual service times are exponentially distributed, the sum of exponential processes is not exponential. To show this, consider the mean and variance. The exponential with a mean of $1/m$ has variance $1/m^2$. The mean of the sum is the sum of the means, so the total mean time is $5/m$.. The variance of the sum is the sum of the variances, so the variance of the overall process in $5/m^2$. However, an exponential process has a variance that is the square of the mean. $5/m^2$ is not the same as $(5/m)^2$.
> [3 pts] If the 5 employees were working in parallel, each with an exponentially distributed completion time with a mean completion of $5/m$, the arrival process at the testing station would be Poisson due to the fact that exponentially distributed service times create Poisson arrival processes and the pooling of Poisson processes is Poisson.

2. Observations of the taxi stand at the Hoboken train station show the following probability mass function. What is the expected number of taxis waiting?

Taxis waiting	Probability
0	.02
1	.10
2	.10
3	.15
4	.30
5	.15
6	.10
7	.05
8	.01
9	.01
10	.01

> The expected value (the mean value) is computed by the formula $\sum_{i=0}^{10} i \cdot p_i = 3.92$

3. The Nawatam DMV office's customer service system is shown by the diagram below. Customers wait in the first queue to be served. A random subset of the customers (10%) are finished after being served by server 1. The rest wait in a second queue to be served by the second server before they can leave. Define an N-tuple that could be use to describe the system state and define the elements.



> [10 pts] This system's state could be defined by the 4-tuple (Q1, S1, Q2, S2), where Q1 is the number of customers waiting in queue 1, S1 is the state of server 1 (busy/idle), Q2 is the number of customers waiting in queue 2, and S2 is the state of server 2 (busy/idle). The number of customers in the customer population is outside the system and does not enter the system state. Likewise, the random selection of customers to wait in queue 2 or return to the customer population is not part of the system state.

4. A system has an exponentially distributed service time with an average service rate of 10 customers per hour. Customers arrive at the system from two exponentially distributed sources. The first generates an arrival rate of 5 customers per hour. The second could generate arrival rates of 2, 4, or 6 customers per hour. Find the expected waiting time in the system for each of the arrival rates.

> [3 points for arrival rates of 2(7) and 4(9). 4 points for the arrival rate of 6(11)]. First, observe that the exponentially distributed arrival rates can be added to get total arrival rates of 7, 9, and 11 customers per hour. Next, realize that for an arrival rate of 11/hour, the arrival rate exceeds the service rate, so the queue grows without bound and the expected waiting time is infinite.
>
> For the other two cases, the expression for the average waiting time is $w = \frac{1}{m - l}$
>
> with m=10 and l=7, w=1/3. For m=10 and l=9, w=1.

5. During a system simulation of a single server queuing system at t=10, the system is in the state (3,1), meaning that 3 customers are in the queue and the server is busy. If the next customer arrival is at t=15, fix the following FEL so that it correctly describes the system.

(Depart, 12) (Depart, 16) (Depart, 14)

> [3 pts]First, the FEL is not in order.
> [4 pts]There are three departures scheduled, there should be 4 – three for customers in the queue and 1 for the customer being served
> [3 pts]The arrival isn't scheduled.
> The FEL should look like this (assuming that a departure at t=18 is added to account for the 4th customer in the system:
>
> (Depart, 12)
> (Depart, 14)
> (Arrive, 15)
> (Depart, 16)
> (Depart, 18)

6. What queueing notation would you use to describe the elevators at first floor in the Howe Center? Assume that the two elevators can carry 15 passengers each, users walk if there are more than 40 people waiting in the lobby, the Stevens staff, guest, and student population equal 5000, the number of resident students is 2000, the elevator services 14 floors, and the elevators return to the lobby at completely random intervals. Make reasonable assumptions about any missing information and state your assumptions.

> [2 points for each parameter] This is an M/G/2/40/5000 queue. We can assume exponential customer arrivals (M). The service time is random (G). There are two servers (elevators). Assume that the number of people who wait in the lobby is the capacity of the system, whether the system or the users enforce the capacity is unimportant – the capacity is 40. Assume that the user population is the number of students, staff and guests, so the user population is 5000. The number of resident students, and the number of floors that the elevator stops on are irrelevant.
>
> Note: The elevator capacity could be used to describe the number of servers or the service times. I'll accept reasonable discussions about how to deal with the elevator capacity. This part of the problem was intentionally left open to see how you deal with non-obvious problems ☺

7. Twenty percent of the time that Seymour Clearly arrives at the 8th Street parking lot for his Monday class, he is able to get a parking space. What is the probability that he will not get parking spot more that 3 times during the 14-week semester?

[10 pts] Consider this as a Bernoulli trial with a sample size of 14 and p=.2.

$$P(X \le 3) = \sum_{x=0}^{3} \binom{14}{x} (.2)^x (.8)^{14-x}$$

where

$$\binom{14}{x} = \frac{14!}{x!(14-x)!}$$

$=(.8)^{14}/0! + (14)*(.2)*(.8)^{13}/1! + (14)(13)*(.2)^2*(.8)^{12}/2! + (14)(13)(12)*(.2)^3*(.8)^{11}/3!$
= .698

There was no data to make any statement about what happened on other days, so it would not be correct to use a sample size of 5*14, to account for all days of the week.

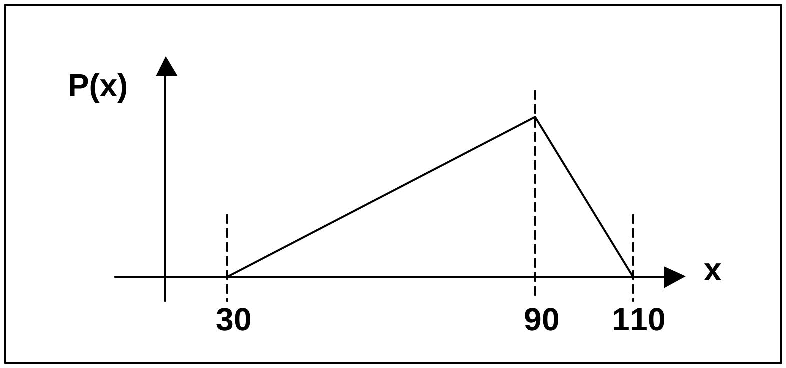
8. Busses arrive at the Hoboken terminal with exponentially distributed interarrival times, with an average of 8 busses arriving per hour. What is the probability that 12 busses will arrive between noon and 1:30 pm? If 6 busses arrived between noon and 1:00 pm, does this alter the probability? What if 10 busses arrived between 11 am and noon?

[6 pts] Since the interarrival process is exponential, the number of busses arriving in a given interval is Poisson. We can calculate the probability directly from $P\left[N(t) = n\right] = \frac{e^{-lt}(l\,t)^n}{n!}$ where n=12, l=8 and t=1.5. In this case, the probability is .114.

If we knew that a given number of busses arrived in a subinterval, we would have to compute the probability for the remainder of the interval – in general, we would expect the probability to change. In this case, the problem becomes, "what is the probability that 6 busses arrive with t=.5, and l=8?" This probability is .104 [2 points for recognizing that the probability changes, 1 extra for computing value].

[2 pts] Events outside the interval are independent, so arrivals between 11 and 12 don't matter.

9. The grade distribution on this midterm is hypothesized to have a triangular distribution which can be graphed as shown below. Find the expected value for the grades. What is the mode?



[3 pts] First, find the peak value of P(x) to be able to describe the triangle. Realize that $\int_{-\infty}^{\infty} P(x)dx = 1$. The peak value, *h*, is 1/40.

[3 pts] Next, express the equation for P(x):

$$P(x) = \begin{matrix} 0 & x \le 30 \\ (x-30)\cdot\frac{1}{40}\cdot\frac{1}{60} & 30 \le x \le 90 \\ (110-x)\cdot\frac{1}{40}\cdot\frac{1}{20} & 90 \le x \le 110 \\ 0 & 90 \le x \end{matrix}$$

[1.5 pts] To find the mean value, calculate

$$\int_{30}^{90}\left[x\cdot(x-30)\cdot\tfrac{1}{40}\cdot\tfrac{1}{60}\right]dx + \int_{90}^{110}\left[x\cdot(110-x)\cdot\tfrac{1}{40}\cdot\tfrac{1}{20}\right]dx$$

[1.5 pts] finding the value of the integral by whatever means:

$$\int_{30}^{90}\frac{x^2}{2400}dx - \int_{30}^{90}\frac{30x}{2400}dx + \int_{90}^{110}\frac{110x}{800}dx - \int_{90}^{110}\frac{x^2}{800}dx =$$

$$\left.\frac{x^3}{3\cdot 2400}\right|_{30}^{90} - \left.\frac{30x^2}{2\cdot 2400}\right|_{30}^{90} + \left.\frac{110x^2}{2\cdot 800}\right|_{90}^{110} - \left.\frac{x^3}{3\cdot 800}\right|_{90}^{110}$$

mean value = 76.66

[1 pt] the mode is the value of x with the highest probability: 90.

10. Given the same overall customer arrival rates and the same average server completion rates, would you expect better average customer waiting time in a pair of parallel M/M/1 queues or a single M/M/2 queue? Why?

[3 pts] Average customer waiting time is better is the M/M/2 queue.

[7 pts] In the parallel M/M/1 queues, the pair of single server queues can individually back up behind a single long service time customer, delaying all the customers waiting in that queue. With two servers attending to a single queue in the M/M/2 case, the customers are less likely to be delayed, since there are two servers, and waiting customers are unlikely to encounter two long service time customers, both consuming server resources at the same time.

[3 pts extra credit] I didn't ask for an analytical explanation, but here is one for extra credit: The average number of customers in the system for the two M/M/1 queues is calculated: If the total customer arrival rate is l, each queue experiences an arrival rate of $l/2$. If every server has a service rate m, the average number of customers in the system is $L = 2\left(\frac{l/2}{m - l/2}\right) = 2\left(\frac{l}{2m - l}\right)$

For the M/M/c case, the queue still sees an overall arrival rate of l. In this case, with c=2, the average number of customers in the system is: $L = \frac{\left(4ml \big/ 2m + l\right)}{2m - l}$ (this is derived by substituting the values for P_0 and r into the expression for L for an M/M/c queue, setting c to 2, and simplifying). Calculating the ratio of L(M/M/1) to L(M/M/2), we find that the ratio of the two is $\frac{\left(2\left(\frac{l}{2m - l}\right)\right)}{\left(\frac{\left(4ml \big/ 2m + l\right)}{2m - l}\right)} = \frac{2m + l}{2m}$. Obviously, this ratio is always greater than 1. This means that there will always be more customers in the system for an M/M/2 system than for an M/M/1 system, with the same arrival rate and server completion times.

11. Print jobs arrive at the Snevets Tech computer center with an exponential distribution having a mean interarrival time of 15 seconds. There are two printers connected to the print server and the print jobs are assigned randomly to the two printers. Both printers have average service times of 20 seconds, but one has a service time that is exponentially distributed, while the other has a service time that is uniformly distributed between 0.01 and 39.99 seconds. Which printer is likely to have the longest average waiting time? What is the average waiting time in the queue for the slower printer?

> [10 pts] Both printers have the same average service time, so the "slower" printer is the one with the larger average waiting time.
>
> By Little's formula ($L_Q = l\, w_Q$), the waiting time can be expressed in terms of the average number of customers in the system and the customer arrival rate, so the waiting time can be expressed in terms of the coefficient of variation for non-exponential distributions, or $w_Q = \frac{L_Q}{l} = \frac{\left(\frac{r^2}{(1-r)}\right)\left(\frac{1+(cv)^2}{2}\right)}{l}$.
>
> Both printers have the same average utilization, $r = \frac{l}{m} = \frac{20}{30} = .6\bar{6}$ (Note – the customer arrivals are equally distributed between the two printers, so each experiences an average interarrival time of 30 seconds.)
>
> The printer with the uniformly distributed service time has a service time variance of 133.33 seconds^2, while the printer with the exponentially distributed service time has a variance of 400 seconds^2, thus the exponential service time printer is the slower one.
>
> The waiting time in the queue for the exponential service time printer is $\frac{r}{m(1-r)}$ or 40 seconds.
>
> [3 extra credit points] For the printer with uniformly distributed service time, the waiting time in the queue is a factor $\frac{1+(cv)^2}{2}$ smaller. In this case, the factor is .667, so the average waiting time in queue for the printer with a uniformly distributed service time is 26.67 seconds.

12. Ignoble Booksellers and Immobile Wireless are the only two stores in a local mall. Ignoble experiences an exponentially distributed customer arrival rate of 20 per hour, while Immobile experiences an exponentially distributed customer arrival rate of 10 per hour. The mall opened at 10 am today and by noon, 25 customers had visited Ignoble, while 40 had visited Immobile. Between noon and 1 pm, which store is expected to receive more customers? How many more?

> [10 pts]The arrival process is memoryless. The only data of relevance is the number of customers per hour. In the hour between noon and 1 pm, 20 customers are expected to visit Ignoble while 10 are expected to visit Immobile. The difference is 10.