EE/CpE 345

Modeling and Simulation

Spring 2003

Class 8

Random-Variate Generation

- Given a process to generate uniformly distributed random numbers, how to generate any arbitrary distribution
  - continuous and discrete valued R.V.s
- Techniques:
  - Inverse Transform
  - Convolution Method
  - Acceptance-Rejection
- Doesn't the simulation environment have the distributions needed?
  - Not always, especially for special distributions
  - It's useful to know how they work to understand constraints

Inverse Transform Technique

- Use exponential distribution to illustrate technique
- Given R_i drawn from *U(0,1)*, generate X_i drawn from exponential distribution
- Most useful when the c.d.f., *F(x)* can be readily inverted

$$f(x) = \begin{cases} l e^{-lx}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 1 - e^{-lx}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\begin{aligned} 1 - e^{-lX} &= R \\ e^{-lX} &= 1 - R \\ -lX &= \ln(1-R) \\ X &= -\frac{1}{l}\ln(1-R) \end{aligned}$$

- Distribution of *R* and *1-R* are identical, allowing simplification

$$X = -\frac{1}{l}\ln(R)$$

Inverse Transform Technique
Exponential Distribution

$N := 200$

$i := 0 .. N - 1$ $\lambda := 1$

$R := \mathrm{runif}(N, 0, 1)$

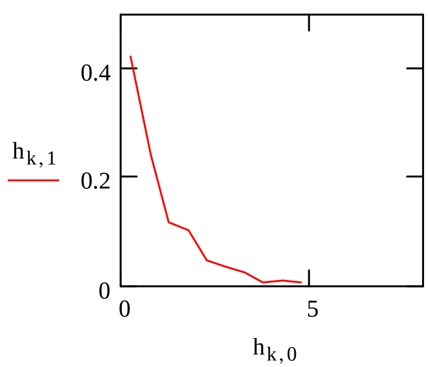
$X := -\left(\frac{1}{\lambda}\right) \cdot \ln(R)$

$M := 10$

$h := \mathrm{histogram}(M, X)$

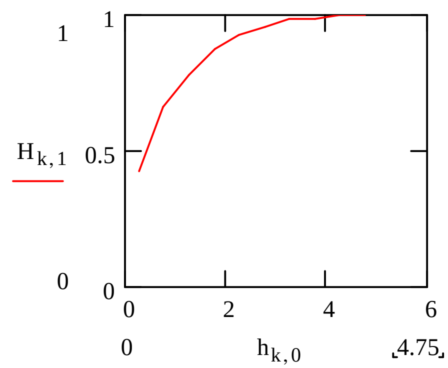
$k := 0 .. M - 1$

$h_{k,1} := \frac{h_{k,1}}{N}$



- Example 8.1: Generate 200 exponentially distributed random numbers, plot histogram

- Generating c.d.f. from histogram:



Verifying that Inverse Transform Technique Generates R.V.s with Correct Distribution

$$P(X_1 \leq x_0) \;=\; P(R_1 \leq F(x_0) \;=\; F(x_0)$$

- X_i is generated from R_i, transformed by *F( )*
- R_i is uniformly distributed on (0,1)

Applying Inverse Transform Technique to Other Distributions

- Weibull

$$f(x)=\begin{cases}\frac{b}{a^b}x^{b-1}e^{-(x/a)^b} & x\ge 0\\ 0 & x<0\end{cases}$$

$$F(x)=\begin{cases}1-e^{-(x/a)^b} & x\ge 0\\ 0 & x<0\end{cases}$$

$$X=a\left[-\ln(R)\right]^{1/b}$$

- Triangular

$$f(x)=\begin{cases}x, & 0\le x\le 1\\ 2-x, & 1<x\le 2\\ 0, & \text{otherwise}\end{cases}$$

$$F(x)=\begin{cases}0, & x\le 0\\ \frac{x^2}{2}, & 0<x\le 1\\ 1-\frac{(2-x)^2}{2}, & 1<x\le 2\\ 1, & x>2\end{cases}$$

$$X=\begin{cases}\sqrt{2R}, & 0\le R\le \frac{1}{2}\\ 2-\sqrt{2(1-R)}, & \frac{1}{2}<R\le 1\end{cases}$$

Examples of Inverse Transform Technique

Weibull Distribution

$N := 2000$

$i := 0..\,N-1$

$\alpha := 1$

$R := \mathrm{runif}(N, 0, 1)$

$\beta := 1.5$

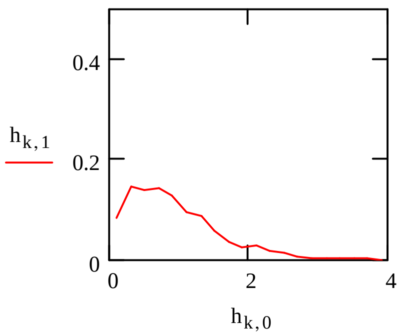
$X := \alpha \cdot (-\ln(R))^{\frac{1}{\beta}}$

$M := 20$

$h := \mathrm{histogram}(M, X)$

$k := 0..\,M-1$

$h_{k,1} := \frac{h_{k,1}}{N}$



Triangular Distribution

$N := 20000$

$i := 0..\,N-1$

$R := \mathrm{runif}(N, 0, 1)$

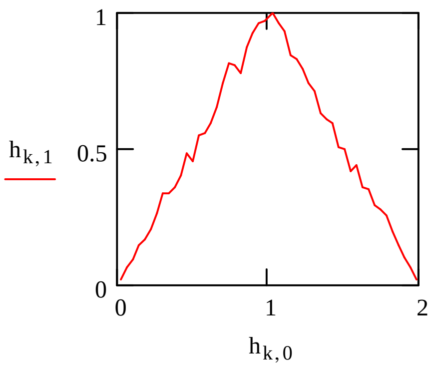
$X_i := \mathrm{if}\left[R_i \le .5, \sqrt{2 \cdot R_i}, 2 - \sqrt{2 \cdot \left(1 - R_i\right)}\right]$

$M := 50$

$h := \mathrm{histogram}(M, X)$

$k := 0..\,M-1$

$h_{k,1} := \frac{h_{k,1} \cdot \frac{M}{2}}{N}$



Empirical Continuous Distributions

- If observed distribution is believed to be discrete, we can use lookup table method previously discussed. What if the distribution is known to be continuous?
  - Interpolate intermediate values: Example 8.2: 5 samples of data are available

i	Interval $x_{(i-1)} < x <= x_{(i)}$	Probability $1/n$	Cumulative Probability, i/n	Slope, a_i
1	0.0 < x <= 0.80	0.2	0.2	4.00
2	0.80 < x <= 1.24	0.2	0.4	2.20
3	1.24 < x <= 1.45	0.2	0.6	1.05
4	1.45 < x <= 1.83	0.2	0.8	1.90
5	1.83 < x <= 2.76	0.2	1.0	4.65

  - Line segment slopes:

$$a_i = \frac{x_{(i)} - x_{(i-1)}}{i/n - (i-1)/n} = \frac{x_{(i)} - x_{(i-1)}}{1/n}$$

  - Inverse c.d.f.:

$$X = x_{(i-1)} + a_i \left(R - \frac{(i-1)}{n} \right) \qquad \text{when} \;\; (i-1)/n < R \leq i/n$$

Empirical Continuous Distributions

- Sometimes a large number of data samples are available to generate empirical distribution
  - not efficient, or always necessary to generate large number of interpolation segments
  - summarize available data into frequency distribution with smaller number of bins
  - fit continuous empirical c.d.f. to frequency distribution

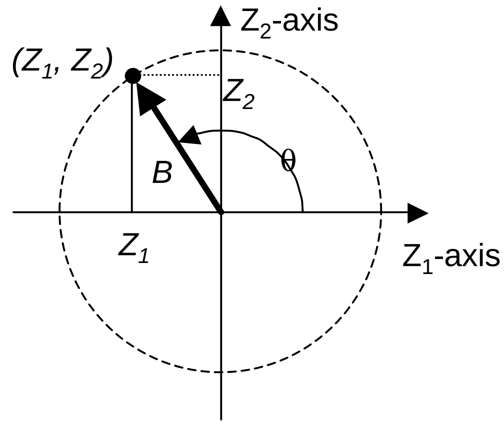
Direct Transform for Normal Distributions

- Normal c.d.f.:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2p}} e^{-\frac{t^2}{2}} dt$$

  – Inverse transform technique cannot be applied - inverse c.d.f. cannot be expressed in closed form.

Direct Transform for Normal Distributions



- Z_1 and Z_2 are normal R.V.s. In polar coordinates:

$$Z_1 = B\cos\theta \qquad B^2 = Z_1^{\,2} + Z_2^{\,2}$$
$$Z_2 = B\sin\theta$$

- B^2 has a chi-square distribution with 2 degrees of freedom - equivalent to an exponential distribution with mean 2. Using Inverse Transform technique:

$$B = \sqrt{-2\ln R}$$

- θ is uniformly distributed on $(0,2\pi)$ and is independent of B.
- Given independent, uniformly distributed R_1 and R_2, generate normal Z_1, Z_2:

$$Z_1 = \sqrt{-2\ln R_1}\cos(2\pi R_2)$$
$$Z_2 = \sqrt{-2\ln R_1}\sin(2\pi R_2)$$

Direct Transform for Normal Distributions

- Computation of sqrt(ln(*R*)) is CPU intensive, but is reused for Z_1 and Z_2
- To obtain normal variates X_i with mean m and variance s^2:

$$X_i = m + s\,Z_i$$

Normal Distribution using Direct Transform

$N := 1000$

$i := 0 .. N - 1$

$R_1 := \mathrm{runif}(N, 0, 1)$

$R_2 := \mathrm{runif}(N, 0, 1)$

$Z_{1_i} := \sqrt{-2 \cdot \ln(R_{1_i})} \cdot \cos(2 \cdot \pi \cdot R_{2_i})$ $\quad \mu := 2$

$Z_{2_i} := \sqrt{-2 \cdot \ln(R_{1_i})} \cdot \sin(2 \cdot \pi \cdot R_{2_i})$ $\quad \sigma := 2.5 \quad \sigma^2 = 6.25$

$X_1 := \mu + \sigma \cdot Z_1$

$X_2 := \mu + \sigma \cdot Z_2$

$M := 50$

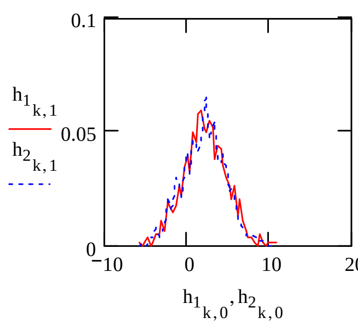
$h_1 := \mathrm{histogram}(M, X_1)$

$h_2 := \mathrm{histogram}(M, X_2)$

$k := 0 .. M - 1$

$h_{1_{k,1}} := \frac{h_{1_{k,1}}}{N} \qquad h_{2_{k,1}} := \frac{h_{2_{k,1}}}{N}$

$\mathrm{mean}(X_1) = 2.092$



$\mathrm{mean}(X_2) = 2.015$

$\mathrm{var}(X_1) = 6.584$

$\mathrm{var}(X_2) = 6.491$

Convolution Method

- Probability distribution of sum of independent R.V.'s is convolution of distributions of variables

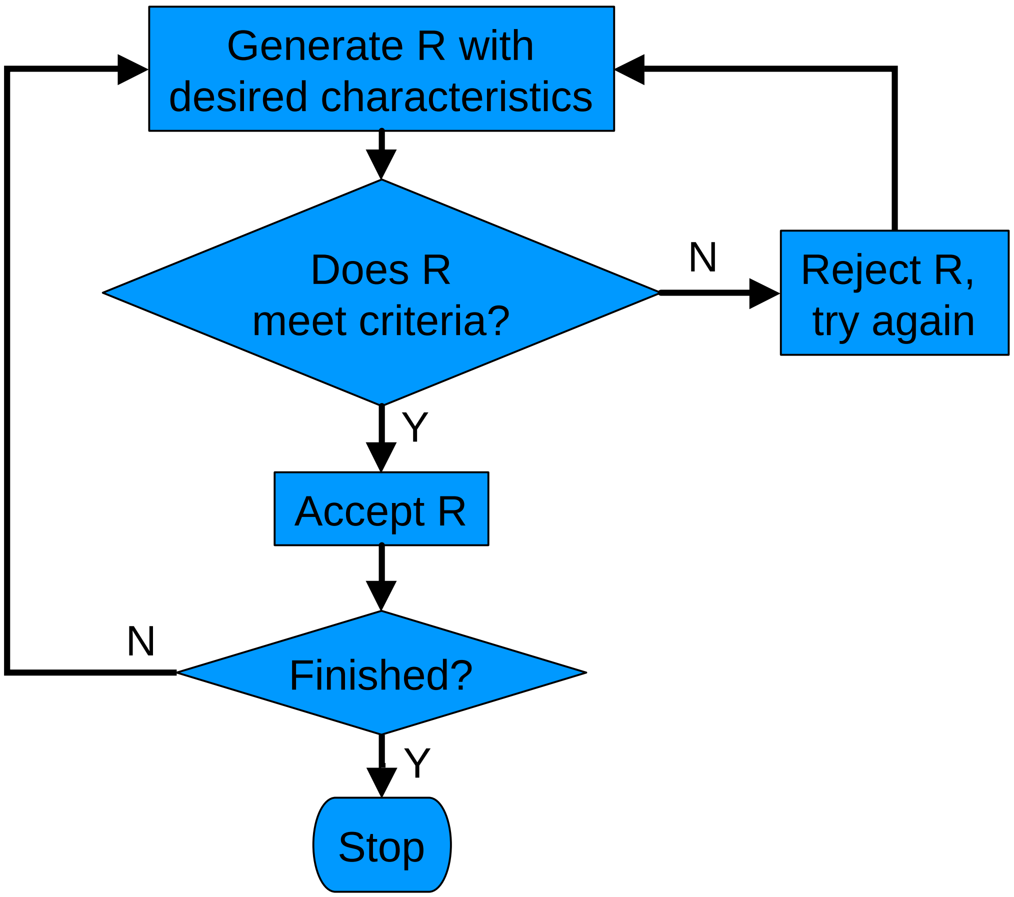
- Erlang R.V. *X* with parameters *(K,q)* is sum of *K* independent exponential R.V.s X_i each with mean *1/Kq*

$$X = \sum_{i=1}^{K} X_i$$

- From Inverse Transform Technique, each X_i is generated with *1/l=1/Kq*

$$\begin{aligned} X &= \sum_{i=1}^{K} -\frac{1}{Kq} \ln R_i \\ &= -\frac{1}{Kq} \ln\left(\prod_{i=1}^{K} R_i \right) \end{aligned}$$

Acceptance-Rejection Technique



- Use the Acceptance-Rejection Technique when other methods have no straightforward solution (e.g., no closed form solution)
- Efficiency depends on fraction of generated random numbers that are rejected

Using Acceptance-Rejection Technique for Poisson Distribution

- Poisson R.V., *N*, with mean a has a probability mass function

$$p(n) = P(N = n) = \frac{e^{-a} a^n}{n!}, \qquad n = 0,1,2...$$

- *N* is the number of arrivals from Poisson arrival process in one unit of time
- Interarrival times $A_1, A_2, \ldots$ of successive customers are exponentially distributed with rate a
- We know how to generate an exponential distribution from Inverse Transform Technique
- Relationship between the discrete Poisson process and continuous exponential distribution:

$$N = n$$

if and only if

$$A_1 + A_2 + ... + A_n \le 1 < A_1 + ... + A_n + A_{n+1}$$

- Generate *n+1* exponential interarrival times until some arrival occurs after *t=1*, then set *N=n*

Using Acceptance-Rejection Technique for Poisson Distribution

- Generate arrival times

$$A_1 + A_2 + \ldots + A_n \le 1 < A_1 + \ldots + A_n + A_{n+1}$$

or $$\sum_{i=1}^{n} -\frac{1}{a} \ln R_i \le 1 < \sum_{i=1}^{n+1} -\frac{1}{a} \ln R_i$$

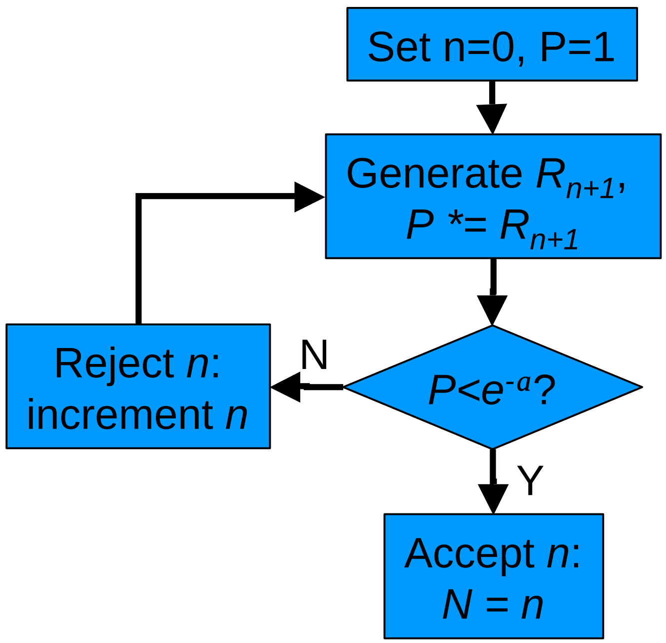
or $$\sum_{i=1}^{n} \ln R_i \ge -a > \sum_{i=1}^{n+1} \ln R_i$$

or $$\ln \prod_{i=1}^{n} R_i \ge -a > \ln \prod_{i=1}^{n+1} R_i$$

or $$\prod_{i=1}^{n} R_i \ge e^{-a} > \prod_{i=1}^{n+1} R_i$$

Using Acceptance-Rejection Technique for Poisson Distribution

- Given a:



Using Acceptance-Rejection Technique for Poisson Distribution

- Example 8.11: Bus with Poisson arrival process, arrival rate a=4 per hour. Generate number of buses arriving during a 1 hour period

- With high acceptance rate, this technique works well. As acceptance rate drops (e.g., for larger a), this technique becomes inefficient.

- An approximation for the Poisson process for a>15 is:

$$Z = \sqrt{-2\ln R_1}\cos(2\pi R_2)$$

$$N = \begin{cases} 0 & \text{if } a + \sqrt{a}\,Z - .5 < 0 \\ \text{ceil}\left(a + \sqrt{a}\,Z - .5\right) & \text{otherwise} \end{cases}$$

$\alpha_{bus} := 4$

$e^{-\alpha_{bus}} = 0.018$

```
N(α) := | n ← 0
        | P ← 1
        | m ← e^(−α)
        | while P ≥ m
        |     | P ← P·rnd(1)
        |     | n ← n + 1
        | return n
```

How many arrivals per hour during a given 8 hour period?

$i := 0..7$

$Arrivals_i := N(\alpha_{bus})$

$$Arrivals = \begin{pmatrix} 3 \\ 4 \\ 7 \\ 3 \\ 4 \\ 2 \\ 4 \\ 3 \end{pmatrix}$$

mean(Arrivals) = 3.75

Homework

- Ch. 8 exercises 6, 24, 28