EE 670 : Homework #3 Solution

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Note: Cover and Thomas, Q. 5.4,5.8,5.14,5.25

1. Question: *Huffman coding.* Consider the random variable:

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.04 & 0.02 \end{pmatrix}$$

   (a) Find a binary Huffman code for X.
   Solution: The Huffman tree for this distribution is:

Codeword								
1	x_1	0.49	0.49	0.49	0.49	0.49	0.51	1
00	x_2	0.26	0.26	0.26	0.26	0.26	0.49	
011	x_3	0.12	0.12	0.12	0.13	0.25		
01000	x_4	0.04	0.05	0.08	0.12			
01001	x_5	0.04	0.04	0.05				
01010	x_6	0.03	0.04					
01011	x_7	0.02						

   (b) Find the expected codelength for this encoding.
   The expected length of the codeword for the binary Huffman code $E(X)$=2.02 bits.

   (c) Find a ternary Huffman code for X.
   Solution:

Codeword					
0	x_1	0.49	0.49	0.49	1
1	x_2	0.26	0.26	0.26	
20	x_3	0.12	0.12	0.25	
22	x_4	0.04	0.09		
210	x_5	0.04	0.04		
211	x_6	0.03			
212	x_7	0.02			

2. Question *Simple optimum compression of a Markov source.* Consider the 3-state Markov process having transition matrix in Table 2: Thus, the probability that S_1 follows S_3 is equal to zero. Design 3 codes C_1, C_2, C_3 (one for each state, S_1, S_2, S_3), each code mapping elements of the set S_i's into sequences of 0's and 1's, such that this Markov process can be sent with maximal compression by the following scheme:

   (a) Note the present symbol S_i.

   (b) Select code C_i.

U_{n-1}/U_n	S_1	S_2	S_3
S_1	1/2	1/4	1/4
S_2	1/4	1/2	1/4
S_3	0	1/2	1/2

Table 1: Markov State Transition Matrix

Next State	S_1	S_2	S_3	
Code C_1	0	10	11	$E(L\|C_1) = 1.5$ bits/symbol
Code C_2	10	0	11	$E(L\|C_1) = 1.5$ bits/symbol
Code C_3	–	0	1	$E(L\|C_1) = 1$ bits/symbol

Table 2: Code

(c) Note the next symbol S_j and send the codeword in C_i corresponding to S_j.

(d) Repeat for the next symbol.

Solution A possible solution is shown in 2: The average message lengths of the next symbol conditioned on the previous state being S_i are just the expected lengths of the codes C_i. Note that this code assignment achieves the conditional entropy lower bound.

To find the unconditional average, we have to find the stationary distribution on the states. Let μ be the stationary distribution. Then:

$$\mu = \mu \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

We can solve this to find that $\mu = (2/9, 4/9, 1/3)$. Thus the unconditional average number of bits per source symbol:

$$\begin{aligned} \mathrm{E}L = & \sum_{i=1}^{3} \mu_i \mathrm{E}(L|C_i) \\ & = \tfrac{2}{9} x 1.5 + \tfrac{4}{9} x 1.5 + \tfrac{1}{3} x 1 \\ & = \tfrac{4}{3} bits/symbol \end{aligned}$$

The entropy rate $\mathcal{H}$ of the Markov chain is:

$$\begin{aligned} \mathcal{H} = & H(X_2|X_1) \\ & = \sum_{i=1}^{3} \mu_I H(X_2|X_1 = S_i) \\ = & 4/3 \ \text{ bits/symbol} \end{aligned}$$

Thus the unconditional average number of bits per source symbol and the entropy rate $\mathcal{H}$ of the Markov chain are equal, because the expected length of each code C_i equals the entropy of the state after state i, $H(X_2|X_1 = S_i$, and thus maximal compression is obtained.

3. Question *Huffman Code* Find the:

   (a) *binary* Huffman code for the random variable X with the probabilities:

   $$p = \left(\frac{1}{21}, \frac{2}{21}, \frac{3}{21}, \frac{5}{21}, \frac{6}{21}\right).$$

   Solution The Huffman tree for this distribution is:

Codeword							
00	x_1	6/21	6/21	6/21	9/21	12/21	1
10	x_2	5/21	5/21	6/21	6/21	9/21	
111	x_3	4/21	4/21	5/21	6/21		
010	x_4	3/21	3/21	4/21			
0110	x_5	2/21	3/21				
0111	x_6	1/21					

   (b) *ternary* Huffman code.
   Solution:

Codeword						
1	x_1	6/21	6/21	6/21	10/21	1
2	x_2	5/21	5/21	6/21		
00	x_3	4/21	4/21	5/21		
01	x_4	3/21	3/21			
020	x_5	2/21	3/21			
021	x_5	1/21				
022	x_5	0/21				

   (c) Calculate $L = \sum p_i \ \ l_i$ in each case.
   Solution The expected length of codewords for the binary Huffman code is $51/21 = 2.43$ bits. The ternary code has an expected length of $34/21 = 1.62$ ternary symbols.

4. Question *Shannon Code.* Consider the following method for generating a code for a random variable X which takes on m values $\{1, 2, \cdots, m\}$ with probabilities $p_1, p_2, \cdots, p_m$. Assume that the probabilities are ordered such that $p_1 \geq p_2 \geq \cdots \geq p_m$. Define $F_i = \sum_{k=1}^{i-1} p_i$, the sum of the probabilities of all symbols less than i. Then the codeword for i is the number $F_i \in [0, 1]$ rounded off to l_i bits, where $l_i = \lceil \log \frac{1}{P_i} \rceil$.

   (a) Show that the code constructed by this process is prefix-free and the average length satisfies:

   $$H(X) \leq L < H(X) + 1,$$

   Solution The Shannon code has $l_i = \lceil \log \frac{1}{p_i} \rceil$, such that we have $\log \frac{1}{p_i} \leq l_i \leq \log \frac{1}{p_i} + 1$, which implies that $H(X) \leq L = \sum p_i l_i x i < H(X) + 1$. To prove that the code is a prefix code, note that by the choice of l_i, we have $2^{-l_i} \leq p_i < 2^{-(l_i-1)}$. Thus $F_j, j > i$ differs from F_i by at least 2^{-l_i}, and will therefore be at least one place different in the first $L - i$ bits of the binary expansion of F_i. Thus the codeword, for $F_j, j > i$ which has length $L_j \geq l_i$ differs from the codeword for F_i at least once in the first l_i places. Thus no codeword is a prefix of any other codeword.

(b) Construct the code for the probability distribution $(0.5, 0.25, 0.125, 0.125)$

Solution We build the following table:

Symbol	Probability	F_i in decimal	F_i in binary	l_i	Codeword
1	0.5	0.0	0.0	1	0
2	0.25	0.5	0.1	2	10
3	0.125	0.75	0.110	3	110
4	0.125	0.875	0.111	3	111

The Shannon code in this case achieves the entropy bound 1.75 bits and is optimal.